Trusses

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1 Introduction

Trusses are structures consisting of straight bars connected at nodes and arranged in triangular patterns. They are the simplest structures to analyse and they have been used many times in the construction of bridges, antennas, space structures, etc. See Fig. 1 for an illustration of *planar* trusses.

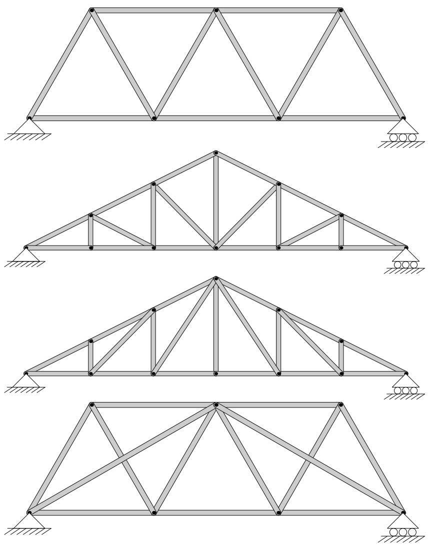

Figure 1: Truss examples.

2 The bar

A bar is a one-dimensional *straight* structural model that can be employed to sustain *axial* loads. In classical structural analysis, the bar carries a axial stress resultant, denoted as N, that is proportional to the axial or longitudinal strain at each of its points.

Let us consider initially a bar of constant cross section A made of an elastic material with Young modulus E and subject to a constant stress resultant N. If the initial and current lengths of the bar are $\ell, \tilde{\ell}$, respectively, we define the axial strain as the scalar

$$\epsilon := \frac{\tilde{\ell} - \ell}{\ell} = \frac{\Delta \ell}{\ell} , \tag{1}$$

and its relation with N is given by the linear relation

$$N = EA\, \epsilon . \tag{2}$$

The *internal energy* of an elastic bar is given by

$$V_{int} = \int_0^{\ell} \frac{1}{2} EA\, \epsilon^2 \, \mathrm{d}x = \frac{1}{2} EA\ell\, \epsilon^2 . \tag{3}$$

3 The bar on the plane

As illustrated in Fig. 1, bars are often placed in tilted configurations that make their analysis slightly more complicated that explained in Section 2. To introduce the modifications we focus on planar trusses, although the extension to three-dimensional problems is straightforward.

Consider the truss of Fig. 2. It connects two nodes with labels 1 and 2, respectively, whose reference positions are the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ and have displacements $\boldsymbol{U}_1, \boldsymbol{U}_2$. The length of the rod is

$$\ell = |\boldsymbol{r}_2 - \boldsymbol{r}_1| , \tag{4}$$

and we define the *unit* vector

$$\boldsymbol{d}_{1,2} := \frac{1}{\ell} (\boldsymbol{r}_2 - \boldsymbol{r}_1) \tag{5}$$

that goes from node 1 to node 2. With these definitions it is immediate to show that the axial strain of the rod can be calculated as

$$\epsilon = \frac{1}{\ell}\left(\boldsymbol{U}_2 - \boldsymbol{U}_1\right)\cdot \boldsymbol{d}_{1,2} \ , \tag{6}$$

where we have used the dot product of vectors on the plane. The axial stress resultant can be calculated using Eq. (25) and the internal energy with Eq. (12).

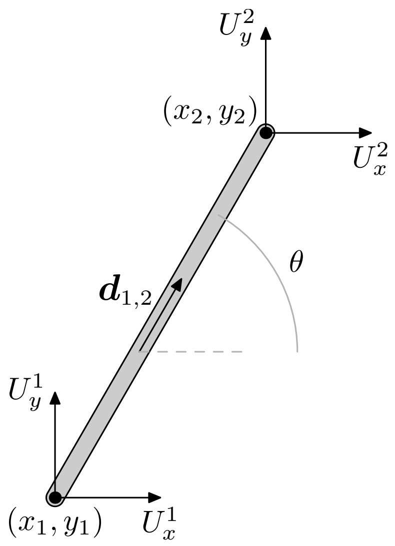

Figure 2: Bar on a plane.

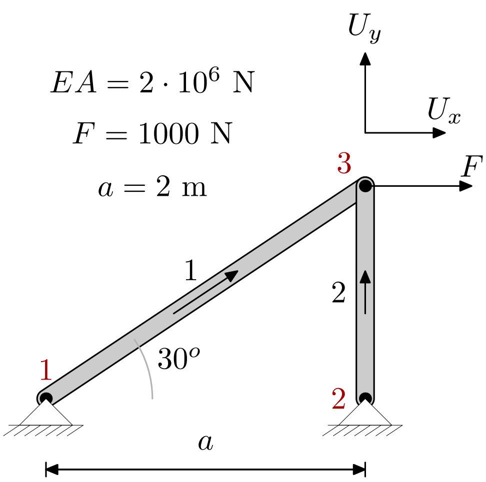

Figure 3: A simple triangular truss.

4 Solution of a planar truss using equilibrium and compatibility

Consider now the simplest planar truss, as depicted in Fig. 3. It consists of two bars pinned at the bottom and connected at node 3. If the truss is subject to a horizontal force F on the free node one could be interested in finding the axial force N in both trusses, the reactions at the supports, as well as the displacement $\boldsymbol{U}$ on node 3. There are several ways of solving this problem and we describe first the so-called *equilibrium* method that is very simple to explain and use in simple trusses such as this one.

To start consider the static equilibrium on all the forces that act upon node 3. If the tilted bar is labeled as 1 and the vertical one as 2, then the equations of horizontal and vertical equilibrium are, respectively,

$$\begin{aligned} N_1 \cos\theta &= F \ , \\ N_1 \sin\theta + N_2 &= 0 \ , \end{aligned} \tag{7}$$

which can be solved to obtain

$$N_1 = F \sec\theta, \qquad F_2 = -F \tan\theta. \tag{8}$$

With this two values we conclude that bar 1 is in traction, and bar 2 is in compression. The strains of the two bars are

$$\epsilon_1 = \frac{N_1}{EA} = \frac{F\sec\theta}{EA} \ , \qquad \epsilon_2 = \frac{N_2}{EA} = -\frac{F\tan\theta}{EA} \ . \tag{9}$$

Once we know the strains in both bars, we can recover the elongation in each of them using Eq. (1) to get

$$\Delta\ell_1 = \epsilon_1 \ell_1 = \frac{F a \sec^2\theta}{EA}, \ \Delta\ell_2 = \epsilon_2 \ell_2 = -\frac{F a \tan^2\theta}{EA}. \tag{10}$$

The vertical displacement of node 3 coincides with $\Delta\ell_2$. The horizontal displacement of this node can only be calculated with some simple, but cumbersome geometrical calculations that give

$$U_x = \Delta\ell_2(\cos\theta + \sin\theta\tan\theta) - \Delta\ell_1 \tan\theta \ . \tag{11}$$

Details are omitted.

Remarks:

a) The reactions on nodes 1 and 2 can be obtained by projecting the stress resultants.

b) This equilibrium method is relatively straightforward because the truss is *statically determinate*. If we were, for example, to connect a third bar from a new support to node 3, the structure will be *statically indeterminate* and its solution, employing equilibrium and compatibility of deformations, would be fairly cumbersome.

c) For all types of trusses, both statically determinate and indeterminate, the calculation of nodal displacements is always complicated. In fact, for the simple example of Fig. 3, the calculations are already non-trivial. For a truss with dozens or hundreds of bars this procedure is too complex to be of practical use.

5 Solution of a planar truss using energy methods

The determination of all the unknown variables of the structure (reactions, strains, axial forces, and displacements) using *energy* methods pivots on the *principle of the minimum potential energy*, which states that the *total* potential energy of the structure is at a minimum when it is in equilibrium. For a truss, this energy is of the form

$$V(\boldsymbol{U}) := \sum_{e=1}^{M} V^e_{int}(\boldsymbol{U}) - \sum_{a=1}^{N} \boldsymbol{F}_a \cdot \boldsymbol{U}_a \ , \tag{12}$$

where e runs through the M bars and a through the N nodes. The term V^e_{int} refers to the internal energy of bar e and is calculated using Eq. (12).

To use this method for the solution of the truss in Fig. 3, we first need to calculate the internal energy of each of the bars as a function of the displacement. Using Eq. (6) we obtain

$$\begin{aligned} \epsilon_1 &= \frac{1}{\ell_1}\boldsymbol{d}_{1,3}\cdot(\boldsymbol{U}_3-\boldsymbol{U}_1) = U_x\frac{\cos\theta}{\ell_1} + U_y\frac{\sin\theta}{\ell_1} \ , \\ \epsilon_2 &= \frac{1}{\ell_2}\boldsymbol{d}_{2,3}\cdot(\boldsymbol{U}_3-\boldsymbol{U}_2) = U_y\frac{1}{\ell_2} \ . \end{aligned} \tag{13}$$

Therefore, the total potential energy of the structure is

$$\begin{aligned} V =& \frac{EA\ell_1}{2}\left(U_x\frac{\cos\theta}{\ell_1} + U_y\frac{\sin\theta}{\ell_1}\right)^2 \\ &+ \frac{EA\ell_2}{2}\left(U_y\frac{1}{\ell_2}\right)^2 - F\,U_x \end{aligned} \tag{14}$$

The equilibrium conditions are given by the relation $\nabla V = \boldsymbol{0}$, that is,

$$\begin{aligned} 0 &= \frac{EA}{\ell_1}(U_x\cos\theta + U_y\sin\theta)\cos\theta - F \ , \\ 0 &= \frac{EA}{\ell_1}(U_x\cos\theta + U_y\sin\theta)\sin\theta + \frac{EA}{\ell_2}U_y \ . \end{aligned} \tag{15}$$

This is a *liner* system of equations. To solve it, let us introduce the stiffness parameters

$$\begin{aligned} K_{xx} &= \frac{EA}{\ell_1}\cos^2\theta \ , \quad K_{xy} = \frac{EA}{\ell_1}\sin\theta\,\cos\theta \ , \\ K_{yy} &= \frac{EA}{\ell_1}\sin^2\theta + \frac{EA}{\ell_2} \ , \end{aligned} \tag{16}$$

that allows us to rewrite Eq. (15) more compactly as

$$\begin{aligned} K_{xx}U_x + K_{xy}U_y &= F \ , \\ K_{xy}U_x + K_{yy}U_y &= 0 \ , \end{aligned} \tag{17}$$

whose solution is

$$U_x = \left(K_{xx} - \frac{K_{xy}^2}{K_{yy}}\right)^{-1} F \ , \qquad U_y = -\frac{K_{xy}}{K_{yy}}U_x \ . \tag{18}$$

Once the displacement of node 3 is known it is straightforward to calculate the strains in the bars using Eq. (13), and from them the axial forces and reactions.

Remarks:

a) The method outlined here is completely general (for linear trusses) and systematic: always the same steps need to be followed and there is no need of complex trigonometric derivations in order to find the complete solution.

b) Whether the structure is statically determinate or indeterminate plays no role at all and the steps to be followed are identical in both cases. In particular, statically indeterminate problems are not more difficult than determinate ones.

6 Matrix formulation

The energy method can be formulated alternatively using matrix algebra, simplifying the computations even further. Let us consider a bar such as the one in Fig. 2. Its strain is

$$\epsilon = \frac{1}{\ell}\begin{bmatrix}-\boldsymbol{d}_{1,2} & \boldsymbol{d}_{1,2}\end{bmatrix}\begin{Bmatrix}\boldsymbol{U}_1\\ \boldsymbol{U}_2\end{Bmatrix} = \boldsymbol{b}\begin{Bmatrix}\boldsymbol{U}_1\\ \boldsymbol{U}_2\end{Bmatrix}, \tag{19}$$

where $\boldsymbol{b}$ is the so-called strain matrix

$$\boldsymbol{b} := \frac{1}{\ell}\begin{bmatrix}-\boldsymbol{d}_{1,2} & \boldsymbol{d}_{1,2}\end{bmatrix}. \tag{20}$$

Introducing the constant $\kappa := EA\ell$ and the stiffness matrix $\boldsymbol{k} := \boldsymbol{b}^T\kappa\,\boldsymbol{b}$ we have that the internal energy of the bar can be written as

$$V_{int} = \frac{1}{2}\boldsymbol{u}^T\,\boldsymbol{b}^T\,\kappa\,\boldsymbol{b}\,\boldsymbol{u} = \frac{1}{2}\boldsymbol{u}^T\,\boldsymbol{k}\,\boldsymbol{u} \ , \tag{21}$$

where $\boldsymbol{u} = \langle \boldsymbol{U}_1 \ \boldsymbol{U}_2\rangle^T$ collects the displacements in the two nodes of the bar. This last expression makes explicit that the internal energy in a (linear and elastic) truss is always a quadratic form.

In a truss that consists of several bars, the total potential energy is thus

$$V = \sum_{e=1}^{M}\frac{1}{2}\boldsymbol{u}_e^T\boldsymbol{k}_e\boldsymbol{u}_e - \sum_{a=1}^{N}\boldsymbol{f}_a\cdot\boldsymbol{U}_a \ . \tag{22}$$

Here, the matrix $\boldsymbol{k}_e$ is the stiffness matrix of bar e and $\boldsymbol{u}_e$ is, as before, the vector collecting the two nodal displacements at the origin and the tip of the same bar.

Global matrix expressions

Using matrix algebra we can define an expression for the energy that is even more compact than Eq. (22). To start, let $\boldsymbol{U}$ be the array that collects the displacements of all the nodes that are allowed to move. Then, let us note that the strain on bar e is $\epsilon_e = \boldsymbol{b}_e\, \boldsymbol{u}_e$ and can be written alternatively as

$$\begin{aligned}\epsilon_e &= \begin{bmatrix}\boldsymbol{b}_e^1 & \boldsymbol{b}_e^2\end{bmatrix}\begin{Bmatrix}\boldsymbol{u}_e^1\\ \boldsymbol{u}_e^2\end{Bmatrix}\\ &= \begin{bmatrix}\boldsymbol{0} & \cdots & \boldsymbol{b}_e^1 & \boldsymbol{0} & \cdots & \boldsymbol{b}_e^2 & \cdots & \boldsymbol{0}\end{bmatrix}\begin{Bmatrix}U_x^1\\ U_y^1\\ U_x^2\\ U_y^2\\ \vdots\\ U_x^N\\ U_y^N\end{Bmatrix}\\ &= \boldsymbol{B}_e\,\boldsymbol{U}\ ,\end{aligned} \tag{23}$$

where now $\boldsymbol{B}_e$ is a large vector full of zeros that contains the two blocks $\boldsymbol{b}_e^1$ and $\boldsymbol{b}_e^2$ of dimensions 1×2 at the appropriate positions. Note that if one or two of the degrees of freedom associated with the block $\boldsymbol{b}_e^i$, with $i = 1$ or 2, is constrained and thus not represented in $\boldsymbol{U}$, the corresponding block or component of $\boldsymbol{b}_e$ should be removed.

Then, all the strains in all the elements can be written compactly as

$$\boldsymbol{E} = \begin{Bmatrix}\epsilon_1\\ \epsilon_2\\ \vdots\\ \epsilon_M\end{Bmatrix} = \begin{bmatrix}\boldsymbol{B}_1\\ \boldsymbol{B}_2\\ \vdots\\ \boldsymbol{B}_M\end{bmatrix}\boldsymbol{U} = \boldsymbol{B}\,\boldsymbol{U}\ . \tag{24}$$

where $\boldsymbol{B}$ is the *global strain matrix*. Last, if we define the constitutive matrix

$$\boldsymbol{C} = \begin{bmatrix}\kappa_1\boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0}\\ \boldsymbol{0} & \kappa_2\boldsymbol{I} & \cdots & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \ddots & \boldsymbol{0}\\ \boldsymbol{0} & \cdots & \cdots & \kappa_M\boldsymbol{I}\end{bmatrix}\ , \tag{25}$$

then we note that the internal energy of the whole truss can be written as

$$V_{int} = \sum_{e=1}^{M}\frac{1}{2}\boldsymbol{u}_e^T\boldsymbol{k}_e\boldsymbol{u}_e = \frac{1}{2}\boldsymbol{U}^T\,\boldsymbol{K}\,\boldsymbol{U} \tag{26}$$

with $\boldsymbol{K}$ being the *global stiffness matrix*

$$\boldsymbol{K} = \boldsymbol{B}^T\,\boldsymbol{C}\,\boldsymbol{B}\ . \tag{27}$$

Let us conclude by noting that if we define the *global force vector* as $\boldsymbol{F} = \langle\boldsymbol{f}_1\ \boldsymbol{f}_2\ \cdots\boldsymbol{f}_N\rangle^T$ then the *total potential energy* of the truss can be compactly written as

$$V = \frac{1}{2}\boldsymbol{U}^T\,\boldsymbol{K}\,\boldsymbol{U} - \boldsymbol{U}^T\,\boldsymbol{F}. \tag{28}$$

Remarks:

a) From Eq. (28) we note that the equilibrium equations, that is, the equations satisfied by the minimizers of V are

$$\boldsymbol{0} = \nabla V(\boldsymbol{U}) = \boldsymbol{K}\,\boldsymbol{U} - \boldsymbol{F}\ . \tag{29}$$

b) The total energy Eq. (28) is a quadratic function of the displacement vector hence it has a unique minimizer.

7 Element-by-element formulation

Expression (28) is the usual form of the energy employed in structural analysis that, in turn, leads to the equilibrium equations (29) that were commonly used in practice when calculations were done by hand. These global expressions have the advantage of being compact, but they are not practical because their construction involves the operations of matrices (notably $\boldsymbol{B}$ and $\boldsymbol{C}$) that are almost empty, and thus much of the operations involve multiplying by zero. There is a slightly different derivation that is more amenable to computer implementation and is often preferred and is described next.

To start, let us consider a typical bar element such as the one in Figure 1. It has length ℓ_e, axial stiffness E_eA_e and connects two nodes that are locally labeled as 1 and 2 (meaning that each bar connects a pair of nodes that are labeled also 1 and 2, but might be different from nodes 1 and 2 of other elements). If the position vectors of these two nodes are referred to as $\boldsymbol{r}_e^1$ and $\boldsymbol{r}_e^2$, respectively, the relative vector that goes from node 1 to 2 and the unit vector in this direction are

$$\boldsymbol{r}_e = \boldsymbol{r}_e^2 - \boldsymbol{r}_e^1\ , \qquad \boldsymbol{d}_e = \frac{1}{\ell_e}\boldsymbol{r}_e\ . \tag{30}$$

With these concepts, the strain of element e can be calculated as

$$\epsilon_e = \boldsymbol{d}_e\cdot\frac{\boldsymbol{u}_e^2 - \boldsymbol{u}_e^1}{\ell_e} = \begin{bmatrix}\boldsymbol{b}_e^1 & \boldsymbol{b}_e^2\end{bmatrix}\begin{Bmatrix}\boldsymbol{u}_e^1\\ \boldsymbol{u}_e^2\end{Bmatrix} = \boldsymbol{b}_e\,\boldsymbol{u}_e. \tag{31}$$

Here $\boldsymbol{u}_e$ is a vector of length 4 that collects the displacements of the two nodes of element e. Once the strain is known, the axial force on the element is

$$N_e = E_e A_e\, \epsilon_e\ . \tag{32}$$

Recall that all the degrees of freedom of the truss can be collected in a "large" vector $\boldsymbol{U}$. This vector, which can be ordered in any arbitrary but fixed way does not hold the displacements of those constrained nodes that are fixed or whose displacement is imposed to a fixed value *a priori*.

The key concept to develop an element-by-element formulation is to note that there exists a map, referred to as `id`, that recovers, for every element e, the index in vector $\boldsymbol{U}$ of the degree of freedom x or y in the *local* node i. More precisely, $U_{\mathtt{id}(e,i,\alpha)}$ is the displacement of the α coordinate (x or y) of the local node i of element e, as placed in the global vector $\boldsymbol{U}$. By convention, if this displacement is constrained, the scalar $U_{\mathtt{id}(e,i,\alpha)}$ is not contained in $\boldsymbol{U}$ but rather set to its known value.

Using this notation, the strain in element e can be calculated as

$$\begin{aligned}
\epsilon_e &= \boldsymbol{b}^e\ \boldsymbol{u}_e \\
&= \begin{bmatrix}\boldsymbol{b}_e^1 & \boldsymbol{b}_e^2\end{bmatrix} \begin{Bmatrix}\boldsymbol{u}_e^1\\ \boldsymbol{u}_e^2\end{Bmatrix} \\
&= \begin{bmatrix}b_{e,x}^1 & b_{e,y}^1 & b_{e,x}^2 & b_{e,y}^2\end{bmatrix} \begin{Bmatrix}U_{\mathtt{id}(e,1,x)}\\ U_{\mathtt{id}(e,1,y)}\\ U_{\mathtt{id}(e,2,x)}\\ U_{\mathtt{id}(e,2,y)}\end{Bmatrix}
\end{aligned} \tag{33}$$

The powerful idea is that, by exploiting the local or global expressions of the strain, all the quantities needed to solve the truss (energy, equilibrium and stiffness) can be calculated in an element-by-element fashion, as we detail next.

Potential energy

The calculation of the potential energy of a truss is simply obtained with the two sums

$$V(\boldsymbol{U}) = \sum_{e=1}^{M} \frac{1}{2} E_e A_e\, \epsilon_e^2\, \ell_e - \sum_{a=1}^{N} \boldsymbol{f}_a \cdot \boldsymbol{U}_a\ , \tag{34}$$

where the strain ϵ_e is calculated, at the element level, as in Eq. (33). The external potential energy has to be computed by accumulating the contribution of each node.

Equilibrium equations

The equilibrium in the truss is given by Eq. (29), expressing that $\boldsymbol{U}$ minimizes the total potential energy. The vector $\boldsymbol{U}$ has length N_f, the number of degrees of freedom and there are as many equilibrium equations of the form

$$\frac{\partial V}{\partial U_a}(\boldsymbol{U}) = 0\ , \tag{35}$$

for $a = 1, 2, \ldots, N_f$. Replacing the expression (34) of the potential energy we obtain

$$\begin{aligned}
0 &= \sum_{e=1}^{M} E_e A_e\, \epsilon_e \frac{\partial \epsilon_e}{\partial U_a} \ell_e - f_a \\
&= \sum_{e=1}^{M} N_e \frac{\partial \epsilon_e}{\partial U_a} \ell_e - f_a .
\end{aligned} \tag{36}$$

The derivative of the element strain can be effected using Eq. (33). For that let us define the Kronecker delta

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Then,

$$\frac{\partial \epsilon_e}{\partial U_a} = \sum_{i=1}^{2} \sum_{\alpha\in\{x,y\}} b_{e,\alpha}^i\, \delta(a, \mathtt{id}(e,i,\alpha)). \tag{38}$$

Hence, the equilibrium equations are

$$f_a = \sum_{e=1}^{M} \sum_{i=1}^{2} \sum_{\alpha\in\{x,y\}} f_{e,\alpha}^i\, \delta(a, \mathtt{id}(e,i,\alpha)), \tag{39}$$

where

$$f_{e,\alpha}^i = N_e\, b_{e,\alpha}^i \ell_e \tag{40}$$

is the internal energy of node e on the local node i in the direction α, whose global label is $\mathtt{id}(e,i,\alpha)$. In practice, to compute all the element contributions to the internal forces, one must go element by element calculating the vector

$$\boldsymbol{f}_e = \begin{Bmatrix} f_{e,x}^1 \\ f_{e,y}^1 \\ f_{e,x}^2 \\ f_{e,y}^2 \end{Bmatrix} = \boldsymbol{b}_e^T N_e\, \ell_e \tag{41}$$

and add each components in the global equation $\mathtt{id}(e,1,x), \mathtt{id}(e,1,y), \mathtt{id}(e,2,x), \mathtt{id}(e,2,y)$, respectively, if their degrees of freedom are constrained. This process is referred to as *assembling*.

Stiffness matrix

The stiffness matrix K is used to solve the global equilibrium equations (29). It is a matrix of dimensions $(2N_f) \times (2N_f)$. The component (a,b) is defined as

$$K_{ab} = \frac{\partial^2 V}{\partial U_a\, \partial U_b} = \frac{\partial f_a}{\partial U_b}. \tag{42}$$

To calculate these scalars first we note that

$$\begin{aligned}\frac{\partial N_e}{\partial U_b} &= E_e A_e \frac{\partial \epsilon_e}{\partial U_b} \\ &= E_e A_e \sum_{j=1}^{2} \sum_{\beta \in \{x,y\}} b^j_{e,\beta}\, \delta(b, \mathtt{id}(e,j,\beta)).\end{aligned} \tag{43}$$

Therefore, using Eqs. (39), (42) and (43) we get

$$K_{ab} = \sum_{e=1}^{M} \sum_{i,j=1}^{2} \sum_{\alpha,\beta \in \{x,y\}} k^{ij}_{e,\alpha\beta} \delta(a, \mathtt{id}(e,i,\alpha)) \\ \delta(b, \mathtt{id}(e,j,\beta)), \tag{44}$$

where

$$k^{i,j}_{e,\alpha\beta} = b^i_{e,\alpha}\, b^j_{e,\beta} E_e A_e \ell_e \tag{45}$$

is the element stiffness of the local node pair (i,j) in the directions (α,β) that corresponds to the two global degrees of freedom with components $(\mathtt{id}(e,i,\alpha), \mathtt{id}(e,j,\beta))$.

As in the case of the equilibrium equations, in practice the local stiffness matrices are computed in an element-by-element fashion as a block of the form

$$\boldsymbol{k}_e = E_e A_e \boldsymbol{b}_e \boldsymbol{b}_e^T \ell_e \ . \tag{46}$$

Once this 4×4 matrix is constructed, each of its components are placed in the correct position of the global stiffness, again by *assembling* them.